Exercise 16

The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

(a) Find the average velocity over each time interval:

(i) $[4, 8]$ (ii) $[6, 8]$

(iii) $[8, 10]$ (iv) $[8, 12]$

(b) Find the instantaneous velocity when $t = 8$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

Solution

Determine the average velocity on each interval of time.

$$\text{(i)} \quad [4,8] \quad v = \frac{s(8)-s(4)}{8-4} = \frac{\left[\frac{1}{2}(8)^2 - 6(8) + 23\right] - \left[\frac{1}{2}(4)^2 - 6(4) + 23\right]}{4} = \frac{(7)-(7)}{4} = 0$$

$$\text{(ii)} \quad [6,8] \quad v = \frac{s(8)-s(6)}{8-6} = \frac{\left[\frac{1}{2}(8)^2 - 6(8) + 23\right] - \left[\frac{1}{2}(6)^2 - 6(6) + 23\right]}{2} = \frac{(7)-(5)}{2} = 1\ \frac{\text{ft}}{\text{s}}$$

$$\text{(iii)} \quad [8,10] \quad v = \frac{s(10)-s(8)}{10-8} = \frac{\left[\frac{1}{2}(10)^2 - 6(10) + 23\right] - \left[\frac{1}{2}(8)^2 - 6(8) + 23\right]}{2} = \frac{(13)-(7)}{2} = 3\ \frac{\text{ft}}{\text{s}}$$

$$\text{(iv)} \quad [8,12] \quad v = \frac{s(12)-s(8)}{12-8} = \frac{\left[\frac{1}{2}(12)^2 - 6(12) + 23\right] - \left[\frac{1}{2}(8)^2 - 6(8) + 23\right]}{4} = \frac{(23)-(7)}{4} = 4\ \frac{\text{ft}}{\text{s}}$$

Calculate the instantaneous velocity.

$$\begin{aligned}
v(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} &= \lim_{h\to 0} \frac{\left[\frac{1}{2}(t+h)^2 - 6(t+h) + 23\right] - \left[\frac{1}{2}t^2 - 6t + 23\right]}{h} \\
&= \lim_{h\to 0} \frac{\left[\frac{1}{2}(t^2 + 2th + h^2) - 6t - 6h + 23\right] - \left[\frac{1}{2}t^2 - 6t + 23\right]}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h} \\
&= \lim_{h\to 0} \frac{th + \frac{1}{2}h^2 - 6h}{h} \\
&= \lim_{h\to 0} \left(t + \frac{1}{2}h - 6\right) \\
&= t - 6
\end{aligned}$$

Therefore, the instantaneous velocity at $t = 8$ is

$$v(8) = 8 - 6 = 2\ \frac{\text{ft}}{\text{s}}.$$

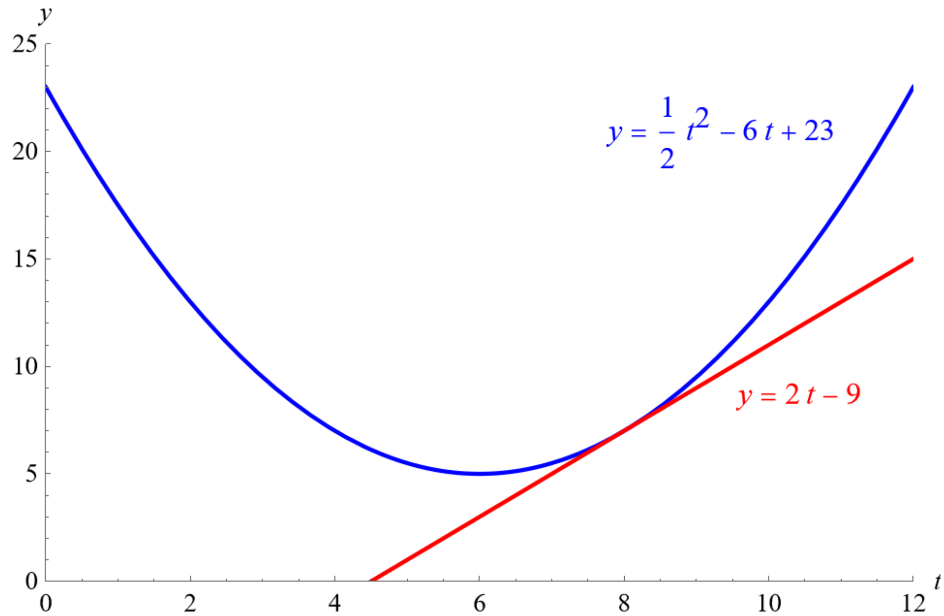
Use the point-slope formula to get the equation of the line with this slope.

$$y - s(8) = 2(t - 8)$$

$$y - 7 = 2t - 16$$

$$y = 2t - 9$$

Below is a plot of the position versus t and the tangent line at $t = 8$.



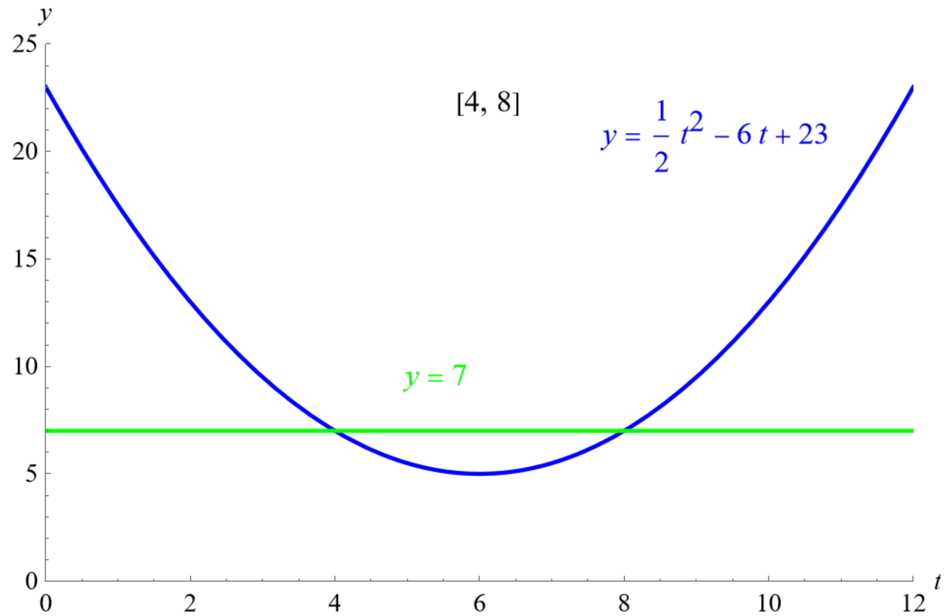
The secant line with slope 0 is

$$y - s(8) = 0(t - 8)$$

$$y - 7 = 0$$

$$y = 7.$$

Below is a plot of the position versus t and the secent line over $[4, 8]$.



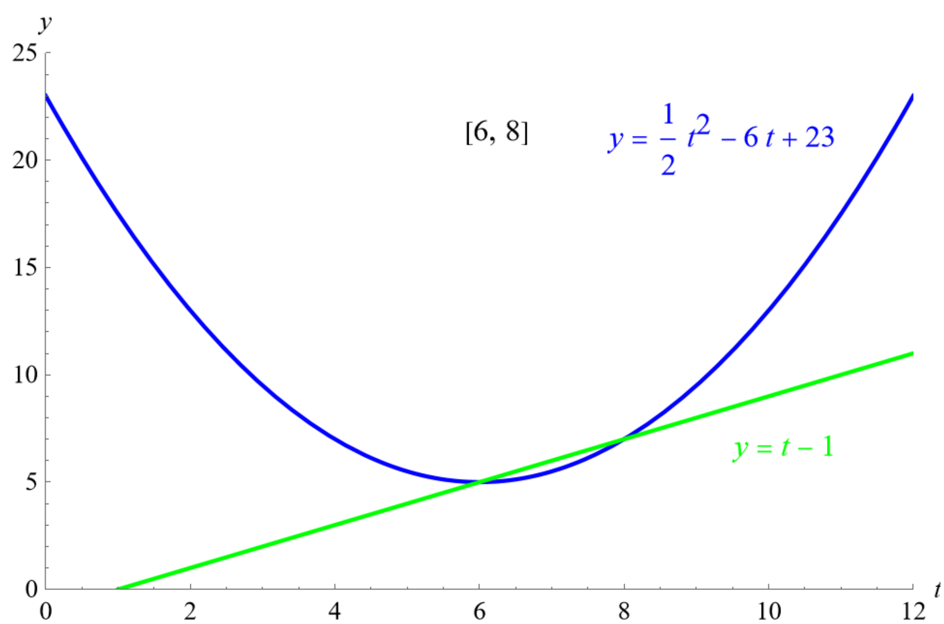
The secant line with slope 1 is

$$y - s(8) = 1(t - 8)$$

$$y - 7 = t - 8$$

$$y = t - 1.$$

Below is a plot of the position versus t and the secent line over $[6, 8]$.



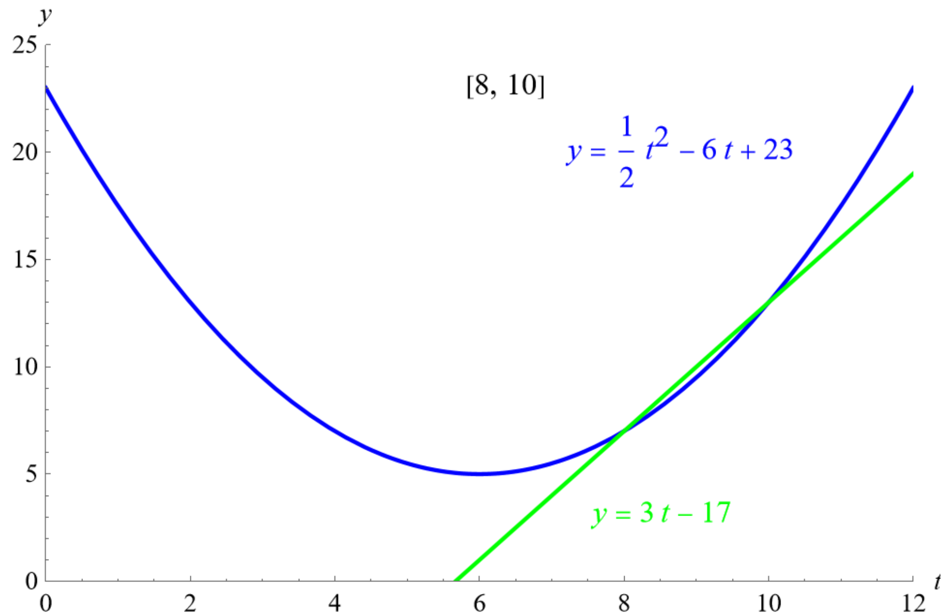
The secant line with slope 3 is

$$y - s(8) = 3(t - 8)$$

$$y - 7 = 3t - 24$$

$$y = 3t - 17.$$

Below is a plot of the position versus t and the secent line over $[8, 10]$.



The secant line with slope 4 is

$$y - s(8) = 4(t - 8)$$

$$y - 7 = 4t - 32$$

$$y = 4t - 25.$$

Below is a plot of the position versus t and the secent line over $[8, 12]$.